Improving the Accuracy of Intermediate-pass Outputs in Multi-pass Real-time Integration Methods

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1. Introduction

In using multi-pass integration methods for real-time simulation of dynamic systems, it may be desirable to employ intermediate-pass state-variable calculations as real-time outputs. For example, the real-time version of 2nd-order Runge-Kutta (RTRK-2) utilizes two evaluations of each state variable per overall integration step. The first evaluation uses Euler integration with a half-step. The second and final evaluation uses Euler integration with a full integration step, but with the state-variable derivative in the formula evaluated at the half-step, using the states evaluated at the half-step as well as any input data points at the half-step. Thus RTRK-2 requires the frame-rate of the input data points to be twice the frame rate of the output data points. However, in real-time simulations there may sometimes be a requirement for input and output data rates to be identical. This can be accomplished by using both half-frame and full-frame data points as real-time outputs in the RTRK-2 simulation. When this is done, it is important to realize that the dynamic errors associated with the half-frame outputs will in general be larger than those of the full-frame outputs, which can lead to undesirable scatter in the dynamic errors of the output data points.

In this paper we introduce formulas which improve the accuracy of the intermediate output data points in multi-pass integration methods. This in turn permits the multi-pass methods to be utilized for real-time simulations with equal input and output data-point frame rates without compromising the dynamic accuracy of the simulation. The multi-pass methods considered include the real-time second-order Runge-Kutta method described above, the third-order Runge-Kutta method (RK-3), and both second and third-order real-time Adams-Moulton predictor-corrector methods (RTAM-2 an RTAM-3).

2. RTRK-2 Integration

Consider the state equation given by

\[ x' = f(x,u(t)) \]  \hspace{1cm} (1)

where \( x \) is a state variable and \( u(t) \) is a time-dependent input. The difference equations for integrating Eq. (1) using RTRK-2 are the following [1]:

\[ x_{n+1/2} = x_n + \frac{1}{2}hf(x_n,u_n) \]  \hspace{1cm} (2)
\[ x_{n+1} = x_n + hf(\hat{x}_{n+1/2},u_{n+1/2}) \]  \hspace{1cm} (3)

Here \( h \) is the integration time step. As an example simulation we consider a second-order system given by the state equations

\[ x' = xdot \]  \hspace{1cm} (4)
\[ xdot' = \omega_n^2(u - x) - 2\xi\omega_n xdot \]  \hspace{1cm} (5)
where \( \omega_n \) is the undamped natural frequency and \( \zeta \) is the damping ratio of the second-order system.

We let the input \( U(t) \) be an acceleration-limited unit-step function given by

\[
U'' = \frac{1}{T^2}, \quad 0 \leq t < T,
\]
\[
= -\frac{1}{T^2}, \quad T \leq t < 2T,
\]
\[
= 0, \quad t \geq 2T. \tag{6}
\]

When integrated twice with \( T \) set equal to \( 1.2/\omega_n \), Eq. (6) results in the input shown in Figure 1. For \( \zeta = 0.25 \), the second-order linear system response to this input with the initial conditions \( x(0) = xdot(0) = 0 \) is also shown in the figure. For comparison purposes the response to a true step input applied at \( t = T \), i.e., at \( \omega_n T = 1.2 \), is shown as the dashed curve in Figure 1. Since the two response curves are quite similar, it is apparent that the use of the acceleration-limited step input, which eliminates discontinuities in input displacement or velocity, does not in fact change appreciably the second-order system step response. Using the acceleration-limited step input eliminates any dynamic simulation errors resulting from discontinuities in input displacement or velocity. This is particularly important when using predictor integration methods of order two and higher, where such discontinuities can be the predominant cause of startup transient errors.

![Figure 1](image)

**Figure 1. Response of second-order system to an acceleration-limited unit-step input.**

In Figure 2 are shown the dynamic errors when RK-2 is used for numerical integration of the second-order system represented by the two first-order differential equations given in Eqs. (4) and (5). For the parameters shown in Figure 1, with the integration step size given by \( h = 0.2 \), Figure 2 shows the simulation errors for both the integer-frame data-point outputs \( x_n \) and
half-integer data-point outputs $\hat{x}_{n+1/2}$. Note that the errors in the half-integer output data points vary noticeably from the errors in the integer output data points. This is due to the larger local truncation error associated with the half-integer data points. We can examine this error by comparing the formula of Eq. (2) for $\hat{x}_{n+1/2}$ with the following Taylor series formulas for the exact half-integer data-point value:

$$ (x_{n+1/2})_{\text{exact}} = x_n + \frac{h}{2} f_n + \frac{h^2}{8} f_n' + \frac{h^3}{48} f_n'' + \cdots, $$

or

$$ (x_{n+1/2})_{\text{exact}} = x_n + \frac{h}{2} f_n + \frac{h^2}{8} f_n' + \frac{|h^2 f_n'|}{|h f_n|} , $$

where

$$ f_n = x_n' = f(x_n, u_n). $$

From Eqs. (2), (8) and (9) the following formula for an improved estimate of each half-integer data point is obtained:

$$ \hat{x}_{n+1/2} = \hat{x}_{n+1/2} + \frac{1}{8} h^2 x_n''. $$

In Eq. (10) we can calculate $x_n''$ using the backward difference approximation $(x_n' - x_n')/h$, which leads to the formula

$$ \hat{x}_{n+1/2} = \hat{x}_{n+1/2} + \frac{h}{8} (x_n' - x_{n-1}'). $$

for the improved estimate of the half-integer data-point output for the $n$th integration frame.

In Figure 3 the results of Figure 2 are repeated, but this time with the half-integer data-point errors calculated using the output $\hat{x}_{n+1/2}$ given by the correction formula of Eq. (11). Comparison with the earlier results of Figure 2 shows that the correction formula has practically eliminated the scatter between integer and half-integer data-point errors. If the half-integer data points are so much more accurate when using Eq. (11) instead of Eq. (2) for the calculation, one might reasonably ask whether the overall simulation accuracy at each integer data point would be significantly improved if the corrected half-integer data point, $\hat{x}_{n+1/2}$, were used instead of $\hat{x}_{n+1/2}$ in Eq. (3) for $x_{n+1}$. The answer is indeed yes, but with the observation that the resulting integration method is in actuality the real time version of the second-order predictor-corrector method[2], i.e. RTAM-2, which has a slightly smaller region of numerical stability in the $\lambda h$ plane, where $\lambda$ represents any eigenvalue associated with a linearized version of the system being simulated. Also, compared with RTRK-2, RTAM-2 has the added disadvantage that the integrated value, $x_{n+1}$, depends on the past value, $x_{n-1}$, of the time derivative as well as the current value, $x_n'$. This in turn can cause substantial additional dynamic error when a discontinuity in the time derivative occurs at $t_n$. We now turn to considering RTAM-2 integration and, in particular, improving the accuracy of the half-integer data-point outputs for this two-pass method.

3. RTAM-2 Integration

The RTAM-2 difference equations for numerical integration of Eq. (1) are given by[2]
Figure 2. Error in integer data-point outputs $x_n$ and half-integer data-point outputs $\hat{x}_{n+1/2}$ using RTRK-2 integration in simulating a second-order system.

Figure 3. Error in integer data-point outputs $x_n$ and corrected half-integer data-point outputs $\hat{x}_{n+1/2}$ based on $x_n'$ and $x_{n-1}'$ using RTRK-2 integration.
\[
\hat{x}_{n+1/2} = x_n + h \left[ \frac{5}{8} f_n - \frac{1}{8} f_{n-1} \right], \quad (12)
\]

\[
x_{n+1} = x_n + hf(\hat{x}_{n+1/2}, u_{n+1/2}). \quad (13)
\]

Figure 4 shows the dynamic errors when the RTAM-2 predictor-corrector method is used for numerical integration of our example second-order system, with the integration step size again given by \( h = 0.2 \). Comparison with Figure 2 shows that the dynamic errors for RTAM-2 are roughly a factor of four smaller than those for RTRK-2. Also, the relative deviation of the half-integer data-point errors from the integer data-point errors is significantly less for RTAM-2. We can follow the same procedure used in the previous section to derive the formulas for improving the half-integer data-point errors. Thus we write the following Taylor series for \( \hat{x}_{n+1/2} \) as given in Eq. (12) by first writing the following Taylor series for \( hf_{n-1} \):

\[
 hf_{n-1} = hf_n - h^2 f_n' + \frac{1}{2} h^3 f_n'' - \frac{1}{6} h^4 f_n''' + \cdots. \quad (14)
\]

Substituting Eq. (14) for \( f_{n-1} \) into Eq. (12), we obtain the following Taylor-series formula for \( \hat{x}_{n+1/2} \):

\[
\hat{x}_{n+1/2} = x_n + \frac{h}{2} f_n + \frac{h^2}{8} f_n' - \frac{h^3}{16} f_n'' + \cdots. \quad (15)
\]

From Eqs. (7) and (15) we can write the formula for an improved estimate for the RTAM-2 half-integer data-point value. This we obtain

\[
\hat{x}_{n+1/2} = \hat{x}_{n+1/2} + \frac{h^3}{12} x_n'''. \quad (16)
\]

Here we can calculate \( x_n''' \) using the backward difference approximation, \((x_n' - 2x_{n-1}' + x_{n-2}')/h^2\), which leads to the formula

\[
\hat{x}_{n+1/2} = \hat{x}_{n+1/2} + \frac{h}{12} (x_n' - 2x_{n-1}' + x_{n-2}') \quad (17)
\]

for the improved estimate of the half-integer data point for the \( n \)th frame of RTAM-2 integration.

In Figure 5 the results of Figure 4 for RTAM-2 integration are repeated, but this time with the half-integer data-point errors calculated using the corrected output \( \hat{x}_{n+1/2} \) given by the formula in Eq. (17). Comparison with the results of Figure 4 shows that the correction formula has all but eliminated the scatter between the integer and half-integer data points. If the half-integer data points are more accurate when using Eq. (17), we might ask whether the overall simulation accuracy at each integer data point would be significantly improved if the corrected half-integer output data-point \( \hat{x}_{n+1/2} \) were used instead of the uncorrected data-point \( \hat{x}_{n+1/2} \) in Eq. (13) for \( x_{n+1/2} \). In this case the answer is no, since the formula for \( \hat{x}_{n+1/2} \) in Eq. (12) is already accurate to order \( h^3 \).

It should be noted that Eq. (16) for the improved estimate, \( \hat{x}_{n+1/2} \), of the half-integer data-point output for RTAM-2 integration was based on the formula for the exact \( x_{n+1/2} \) to order \( h^3 \). Thus Eq. (16) will produce an improved estimate which exhibits zero local truncation error to order \( h^3 \). But the RTAM-2 integration method produces a local truncation error given approximately by[2]
Figure 4. Error in integer data-point outputs $x_n$ and half-integer data-point outputs $\hat{x}_{n+1/2}$ using RTAM-2 integration in simulating a second-order system.

Figure 5. Error in integer data-point outputs $x_n$ and corrected half-integer data-point outputs $\hat{x}_{n+1/2}$ based on $x'_n$, $x'_{n-1}$ and $x'_{n-2}$ using RTAM-2 integration.

\[(x_{n+1} - x_n) - (x_{n+1} - x_n)^{\text{exact}} = -\frac{1}{24} h^3 x''_{n}.\]  

(18)
For the error in the half-integer data-point \( x_{n+1/2} \) to fall half-way between the error in \( x_{n+1} \) and \( x_n \), it follows that the local truncation error in \( x_{n+1/2} \) should be one-half the local truncation error in \( x_{n+1} \), as given in Eq. (18), i.e., \( -(1/48)h^3x'''_n \). When this error is added to \( \hat{x}_{n+1/2} \), as given in Eq. (17), it results in the following formula for an improved estimate for the half-integer data point:

\[
\tilde{x}_{n+1/2} = \hat{x}_{n+1/2} + \frac{h}{16}(x'_n - 2x'_{n-1} + x'_{n-2}) .
\]  

(19)

Figure 6 shows the simulation errors when the half-integer data points are calculated using Eq. (19). Comparison with the previous results in Fig. 5 clearly shows that the scatter between integer and half-integer data-point errors has been further eliminated.

### Figure 6. Error in integer data-point outputs \( x_n \) and corrected half-integer data-point outputs \( \tilde{x}_{n+1/2} \) based on \( x'_n, x'_{n-1} \) and \( x'_{n-2} \) using RTAM-2 integration.

#### 4. RK-3 Integration

We now turn to consideration of third-order integration methods. We first examine RK-3 (third-order Runge-Kutta) integration. For numerical integration of Eq. (1) this three-pass method utilizes the following difference equations[1]:

\[
\hat{x}_{n+1/3} = x_n + \frac{1}{3}hf(x_n, u_n) ,
\]  

(20)

\[
\hat{x}_{n+2/3} = x_n + \frac{2}{3}hf(\hat{x}_{n+1/3}, u_{n+1/3}) ,
\]  

(21)

\[
x_{n+1} = x_n + \frac{1}{4}hf(x_n, u_n) + \frac{3}{4}hf(\hat{x}_{n+2/3}, u_{n+2/3}) .
\]  

(22)
Because the input data points \( u_n, u_{n+1/3} \) and \( u_{n+2/3} \) are available in real time at the start of the first, second and third passes, respectively, through the state equations, this RK-3 algorithm represents a real-time integration method. The first integration pass of Eq. (20) is Euler integration with a step size of \( h/3 \). The second pass of Eq. (21) is Euler integration with a step size of \( 2h/3 \), with the derivative evaluated half-way through the step. The third pass of Eq. (22) produces an integrated output data-point \( x_{n+1} \) with a local truncation error proportional to \( h^4 \). This in turn produces a global truncation error proportional to \( h^3 \).

Figure 7 shows the dynamic errors when the RK-3 method is used for numerical integration of our example second-order system, with the integration step size given by \( h = 0.3 \) instead of 0.2 to reflect the need for three instead of two passes through the state equations for each overall integration step. Comparison with Figure 2 shows that the dynamic errors of the integer data points \( x_n \) are an order of magnitude smaller than those for RTRK-2. But Figure 7 also shows that the intermediate data-point errors in \( \hat{x}_{n+1/3} \) and \( \hat{x}_{n+2/3} \) for the RK-3 simulation exhibit much larger deviations from the integer data-point errors in \( x_n \).

To derive formulas for calculating improved estimates of the intermediate data points for RK-3 integration, we again start by writing Taylor-series formulas for the exact intermediate data points. For \( x_{n+1/3} \) the series is given by

\[
(x_{n+1/3})_{\text{exact}} = x_n + \frac{h}{3} x''_n + \frac{h^2}{18} x'''_n + \frac{h^3}{162} x''''_n + \ldots .
\]  

From Eqs. (20) and (23) we obtain the following formula for an improved estimate of \( x_{n+1/3} \):
\[ \hat{x}_{n+1/3} = \hat{x}_{n+1/3} + \frac{h^2}{18} x_n'' + \frac{h^3}{162} x_n'''. \]  

(24)

For the exact intermediate data-point \(x_{n+2/3}\) the Taylor series is given by

\[ (x_{n+2/3})_{\text{exact}} = x_n + \frac{2}{3} h x_n' + \frac{2}{9} h^2 x_n'' + \frac{4}{81} h^3 x_n''' + \ldots . \]  

(25)

We can rewrite Eq. (21) as

\[ \hat{x}_{n+2/3} = x_n + \frac{2}{3} h x_n' + \left(\frac{\hat{x}'}{3} \right)x_n''. \]  

(26)

where, from Eq. (20)

\[ \hat{x}'_{n+1/3} = x_n' + \frac{h}{3} x_n''. \]  

(27)

Substituting \(\hat{x}'_{n+1/3}\) from Eq. (27) into Eq. (26), we obtain

\[ \hat{x}_{n+2/3} = x_n + \frac{2}{3} h x_n' + \frac{2}{9} h^2 x_n''. \]  

(28)

From Eqs. (25) and (28) we can write the following formula for the improved estimate of \(x_{n+2/3}\):

\[ \hat{x}_{n+2/3} = \hat{x}_{n+2/3} + \frac{4}{81} h^3 x_n'''. \]  

(29)

To calculate numerical approximations for the derivative terms \(h^2 x_n''\) and \(h^3 x_n'''\) in Eqs. (24) and (29), we use the data points \(x_n', x_{n-1/3}'\) and \(x_n'\). We begin by writing Taylor series expansions for \(x_{n-1/3}'\) and \(x_n'\) about \(x_n'\), including terms in \(h x_n''\) and \(h^2 x_n'''\). From the two resulting equations we can determine formulas for \(h x_n''\) and \(h^2 x_n'''\). Thus we obtain

\[ h x_n'' = 4 x_n' - \frac{9}{2} x_{n-1/3}' + \frac{1}{2} x_{n-1}', \quad h^2 x_n''' = 6 x_n' - 9 x_{n-1/3}' + 3 x_{n-1}'. \]  

(30)

Substituting \(h x_n''\) and \(h^2 x_n'''\) from Eq. (30) into Eqs. (24) and (29), we obtain the following formulas for the improved estimates of the RK-3 data points \(x_{n+1/3}\) and \(x_{n+2/3}\):

\[ \hat{x}_{n+1/3} = \hat{x}_{n+1/3} + h \left[ \frac{7}{27} x_n' - \frac{11}{36} x_{n-1/3}' + \frac{5}{108} x_{n-1}' \right], \]  

(31)

\[ \hat{x}_{n+2/3} = \hat{x}_{n+2/3} + h \left[ \frac{8}{27} x_n' - \frac{4}{9} x_{n-1/3}' + \frac{4}{27} x_{n-1}' \right]. \]  

(32)

In Figure 8 the results of Figure 7 for RK-3 integration are repeated, but this time with the corrected intermediate data points \(\hat{x}_{n+1/3}\) and \(\hat{x}_{n+2/3}\) as calculated using Eqs. (31) and (32). Comparison with the previous results in Figure 7 shows that the scatter in the intermediate data points has again almost been eliminated.

5. RTAM-3 Integration

Finally we consider RTAM-3 integration. The difference equations for numerical integration are given by[2]

\[ \hat{x}_{n+1/2} = x_n + h \left[ \frac{17}{24} f_n - \frac{7}{24} f_{n-1} + \frac{1}{12} f_{n-2} \right], \]  

(33)
Figure 8. Error in integer data-point outputs $x_n$ and corrected data-point outputs $\hat{x}_{n+1/3}$ and $\hat{x}_{n+2/3}$ based on $x_n$, $x'_{n-1/3}$ and $x'_{n-1}$ using RK-3 integration.

$$x_{n+1} = x_n + h \left[ \frac{10}{9} f(\hat{x}_{n+1/2}, u_{n+1/2}) - \frac{1}{6} f_n + \frac{1}{18} f_{n-1} \right].$$  (34)

Here Eq. (33) represents a third-order predictor method for determining $\hat{x}_{n+1/2}$ and Eq. (34) is a third-order corrector method for calculating $x_{n+1}$ based on $f_{n+1/2}, f_n$ and $f_{n-1}$.

Because RTAM-3 is a two-pass integration method, we utilize a step size given by $h = 0.2$ in simulating our second-order system. Figure 9 shows the resulting dynamic errors. Comparison with the RK-3 integer data-point errors in Figure 7 indicates that RTAM-3 is more than twice as accurate. Note also that the intermediate half-integer data-point errors for RTAM-3 in Figure 9 exhibit much less deviation from the integer data-point errors than is apparent in Figure 7 for RK-3. This is because the integration formula in Eq. (33) used to calculate the half-integer data-point for RTAM-3 is a third-order method, compared with the first and second-order methods used in Eqs. (20) and (21) for the RK-3 intermediate data points.

To derive the formula for an improved estimate of $x_{n+1/2}$ for RTAM-3 integration, we note that

$$f_{n-2} = f_n - 2hf'_n + 2h^2f''_n - \frac{4}{3} h^3f'''_n + \cdots.$$  (35)

Substituting $f_{n-1}$ and $f_{n-2}$ from Eqs. (14) and (35) into Eq. (33), we obtain the following formula for the Taylor-series representation of $\hat{x}_{n+1/2}$:

$$\hat{x}_{n+1/2} = x_n + h\frac{1}{2} x'_n + \frac{h^2}{8} x''_n + \frac{h^3}{48} x'''_n - \frac{h^4}{16} x''''_n + \cdots.$$  (36)
Figure 9. Error in integer data-point outputs $x_n$ and half-integer data-point outputs $\hat{x}_{n+1/2}$ using RTAM-3 integration in simulating a second-order system.

The Taylor series formula for the exact intermediate data-point $x_{n+1/2}$ is given by

$$ (x_{n+1/2})_{\text{exact}} = x_n + \frac{h}{2} x'_n + \frac{h^2}{8} x''_n + \frac{h^3}{48} x'''_n + \cdots. \quad (37) $$

From Eqs. (36) and (37) we can rewrite the formula for $(x_{n+1/2})_{\text{exact}}$ as

$$ (x_{n+1/2})_{\text{exact}} = \hat{x}_{n+1/2} + \frac{25}{384} h^4 x'''_n. \quad (38) $$

We also note that the local truncation error for RTAM-3 integration is given by

$$ (x_{n+1} - x_n) - (x_{n+1} - x_n)_{\text{exact}} = -\frac{1}{36} h^4 x'''_n. \quad (39) $$

Following the same procedure used to derive Eq.(10) for the improved estimate $\hat{x}_{n+1/2}$ for the half-integer data point in the case of RTAM-2, we add one-half of the local truncation error given by Eq. (39) for $x_{n+1}$ to Eq. (38) for $(x_{n+1/2})_{\text{exact}}$ to obtain the formula for the improved estimate, $\tilde{x}_{n+1/2}$. Thus we obtain

$$ \hat{x}_{n+1/2} = \hat{x}_{n+1/2} + \frac{25}{384} h^4 x'''_n \quad (40) $$

To calculate $x'''_n$ we use the following approximation:

$$ x'''_n = \frac{x'_n - 3x'_{n-1/2} + 3x'_{n-1} - x'_{n-3/2}}{(h/2)^3}. \quad (41) $$
In Figure 10 the dynamic errors when using RTAM-3 are again shown, but this time with the intermediate data points calculated using Eqs. (40) and (41). Comparison with the previous results in Figure 9 shows that the utilization of Eq. (40) results in a significant improvement in the scatter between integer and half-integer data-point errors.

Further improvement in the accuracy of intermediate data points can be achieved when both $x_n$ and $x'_n$ are state variables in a simulation. This will always be the case when an acceleration $x''_n$ is integrated twice to obtain both a velocity $x'_n$ and a displacement $x_n$. It is also true for the second-order system given in Eqs. (4) and (5) and used for all the example simulations thus far in this report. For example, consider RTAM-3. To calculate the data-point $\hat{x}_{n+1/2}''$ in the first pass through the state equations, the acceleration $x''_n$ must be calculated and therefore is available, along with $\hat{x}_{n-1}''$ and $\hat{x}_{n-2}''$ to compute the following backward-difference approximation for $h^4 x'''_n$:

$$h^4 x'''_n = h^2 (x''_n - 2 \hat{x}_{n-1}'' + x_{n-2}'') . \tag{42}$$

This formula is then substituted into Eq. (40) to compute the half-integer data points. Figure 11 shows the resulting dynamic errors when RTAM-3 is again used for the simulation. Comparison with the previous results in Figure 10 shows that utilization of the more accurate formula of Eq. (42) has all but eliminated the scatter between integer and half-integer data-point errors.

It should be noted that the utilization of Eq. (42) instead of (41) provides not only a more accurate calculation of $x'''_n$, but also a calculation more robust with respect to the discontinuities in the acceleration-limited unit step input of Eq. (6). This is because Eq.(42) utilizes $x''_n$ in the computation of $x'''_n$. 

Figure 10. Error in integer data-point outputs $x_n$ and corrected half-integer data-point outputs $\hat{x}_{n+1/2}$ based on $x'_n, x'_{n-1/2}, x''_{n-1}$ and $x''_{n-3/2}$ using RTAM-3 integration.
Figure 11. Error in integer data-point outputs $x_n$ and corrected half-integer data-point outputs $\hat{x}_{n+1/2}$ based on $x''_n$, $x''_{n-1}$ and $x''_{n-2}$ using RTAM-3 integration.

7. Summary

We have developed formulas for improving the dynamic accuracy of intermediate data-point outputs when utilizing multi-pass integration methods for real-time simulation, RTRK-2 and RTAM-2 in the case of second-order methods, RK-3 and RTAM-3 in the case of third-order methods. Application of these formulas makes the use of multi-pass methods practical for real-time simulations that require the use of equal input and output data-sequence frame rates. To illustrate the effectiveness of the formulas for correcting the accuracy of intermediate data-point outputs, we have examined the response of a second-order system to an acceleration-limited unit-step input.

It should be noted that the intermediate data-point correction formulas introduced here are based on utilizing the first time derivatives of the output state-variables. We have also shown that intermediate data-point correction formulas based on the second time derivatives of the output state-variables result in more accurate intermediate data-point correction formulas. These formulas can be used when simulating a dynamic system for which both the output and output time derivative are state variables. There may be cases for which the time-derivative of the output state-variable is actually required as a real-time output in a simulation. Then the formulas introduced here can also be used to correct the intermediate data-points representing output time derivatives. However, for this case we have found that the correction formulas can be significantly less accurate. Also, when applied to real-time simulation of first-order dynamic systems, the correction formulas for intermediate output data-points may in general provide less accurate results.
8. References
