Predictor Methods in Real-time Simulation

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1. Introduction

In real-time simulation it may sometimes be necessary to advance a data sequence \( \{x_n\} \) by a given time interval in order to compensate for a delay of the same time interval which occurs somewhere else in the simulation. For example, it is well known that a D to A (digital-to-analog) converter that employs a zero-order hold introduces an effective delay equal to one-half the time step \( h \) associated with the data sequence driving the D to A converter. If left uncompensated, this can introduce a significant dynamic error in a closed-loop, real-time simulation. Additional unwanted delays may also occur in the interface used to transfer data between simulation processors, or in data transfers between simulation processors and actual hardware in a hardware-in-the-loop simulation.

In this paper we present a number of formulas for calculating an estimated data sequence \( \hat{x}_{n+p} \) from a given data sequence \( x_n \), where \( h \) is the data sequence time step and \( ph \) is the prediction time interval. For each formula we will derive an equation for the approximate error in the predicted data point \( \hat{x}_{n+p} \) which, in turn, will allow us to compare the effectiveness of different predictor formulas.

We begin by considering the Taylor series for the data point \( x_{n+p} \) in terms of \( x_n \) and its time derivatives. Thus we can write

\[
x_{n+p} = x_n + ph x'_n + \frac{1}{2} (ph)^2 x''_n + \frac{1}{6} (ph)^3 x'''_n + \cdots + \frac{1}{k!} (ph)^k \frac{d^k x}{dt^k} + \cdots .
\]  

Equations for the estimated data-point \( \hat{x}_{n+p} \) are determined by deriving formulas for approximations of the time derivatives of \( x \) in Eq. (1) in terms of current and past values of \( x_n \) and its time derivatives.

2. Predictors Using \( x_n, x_{n-1}, x_{n-2}, \ldots \)

We first consider formulas for converting a data sequence \( \{x_n\} \) to a data sequence \( \{x_{n+p}\} \), i.e., predicting data sequences \( ph \) seconds into the future, using \( x_n, x_{n-1}, x_{n-2}, \ldots \). We begin by writing the following Taylor series formulas for \( x_n, x_{n-1}, x_{n-2}, \ldots \), where \( x'_n = f'_n \):  

\[
x_{n-1} = x_n - hf_n + \frac{1}{2} h^2 f'_n - \frac{1}{6} h^3 f''_n + \frac{1}{24} h^4 f'''_n - \cdots ,
\]  

\[
x_{n-2} = x_n - 2hf_n + 2h^2 f'_n - \frac{4}{3} h^3 f''_n + \frac{2}{3} h^4 f'''_n - \cdots ,
\]  

\[
x_{n-3} = x_n - 3hf_n + \frac{9}{2} h^2 f'_n - \frac{9}{2} h^3 f''_n + \frac{27}{8} h^4 f'''_n - \cdots ,
\]  

\[
x_{n-4} = x_n - 4hf_n + 8h^2 f'_n - \frac{32}{3} h^3 f''_n + \frac{32}{3} h^4 f'''_n - \cdots ,
\]  

\[ \vdots \]

To derive a predictor formula based on \( x_n, x_{n-1} \) and \( x_{n-2} \), we solve Eqs. (2) and (3) for \( f_n \) and \( f'_n \) to order \( h^3 \). Thus we obtain
\[ h f_n = \frac{3}{2} x_n - 2 x_{n-1} + \frac{1}{2} x_{n-2} + \frac{1}{3} h f''_n, \quad (6) \]
\[ h f'_n = x_n - 2 x_{n-1} + x_{n-2} + h f''_n. \quad (7) \]

The formulas for the estimated first and second derivatives are then:
\[ h \hat{f}_n = \frac{3}{2} x_n - 2 x_{n-1} + \frac{1}{2} x_{n-2}, \quad (8) \]
\[ h \hat{f}'_n = x_n - 2 x_{n-1} + x_{n-2}, \quad (9) \]

with the errors in the formulas given by
\[ h \hat{f}_n - h f_n = -\frac{1}{3} h f''_n, \quad (10) \]
\[ h \hat{f}'_n - h f'_n = -h f''_n. \quad (11) \]

The prediction formula for \( \hat{x}_{n+p} \) becomes
\[ \hat{x}_{n+p} = x_n + ph \hat{f}_n + \frac{1}{2} (ph)^2 \hat{f}'_n. \quad (12) \]

From Eqs. (1), (10), (11) and (12) it follows that the predictor error is given approximately by
\[ \hat{x}_{n+p} - x_{n+p} = -\left[ \frac{1}{3} p + \frac{1}{2} p^2 + \frac{1}{6} p^3 \right] h f''_n. \quad (13) \]

For an example, assume that the data-sequence \( \{x_n\} \) represents the output response of a second-order linear system with undamped natural frequency given by \( \omega_n \) and damping ratio \( \xi \) given by 0.25. For a prediction interval of \( 4h \), i.e., \( p = 4 \), and a data-sequence time-step \( h = 0.2/\omega_n \), Figure 1 shows the predictor response \( \hat{x}_{n+p} \) compared with \( (x_n+p)_{exact} \), the ideal response. Note that the ideal predictor response starts at \( \omega_n t = -ph = -4(0.2) = -0.8 \), whereas the actual predictor re-sponse \( \hat{x}_{n+p} \) doesn't start until \( \omega_n t > 0 \). This is because the predictor formulas represented above by Eqs. (8) through (13) cannot provide a response which ideally should occur prior to the input data stimulus at \( t = 0 \). For \( t > 0 \), Figure 1 shows that the predictor data-sequence output \( \hat{x}_{n+p} \) exhibits a reasonably close match with the ideal predictor output \( (x_{n+p})_{exact} \), despite the relatively large four-step prediction interval represented by \( p = 4 \). By comparison, consider the predictor error when the data sequence \( \{\tilde{x}_n\} \) is a sinusoid with \( \tilde{x}_n = \sin(\omega_n t) \). In this case \( f'' \) in Eq. (13) is equal to \( -\omega^2 \cos(\omega_n t) \). Then Eq. (13) indicates that the error in predictor output data-point \( \hat{x}_{n+p} \) is given approximately by \( (p^3 + p^{3/2} + p^{3/6})(\omega h)^3 \cos(\omega_n t) \). For \( p = 4 \) and \( \omega h = 0.2 \), which represent parameters equivalent to those used in Figure 1, the steady-state error represented by Eq. (13) is a sinusoid with an amplitude equal to 0.16. Except for the startup error at \( t = 0 \) the errors for the predictor example shown in Figure 1 are smaller in magnitude than 0.16.

Figure 2 repeats the example shown in Figure 1, but with half the data-sequence step size, i.e., \( \omega_n h = 0.1 \) instead of 0.2. In order to maintain the same predictor lead time \( t_p = ph \) when the step size is halved, we increase the lead-time index \( p \) from 4 to 8 in obtaining the results shown in Figure 2. Note that the predictor errors in Figure 2 show only a modest decrease compared with the errors in Figure 1. This is not surprising, since for \( p = 8 \) and \( \omega_n h = 0.1 \), Eq. (13) yields an error in \( \hat{x}_{n+p} \) which is only three quarters of the error for for \( p = 4 \) and \( \omega_n h = 0.2 \).
Figure 1. Predictor output \( \hat{x}_{n+p} \) for input \( x_n \) representing the unit-step response of a 2nd-order system; predictor output formula based on \( x_n, x_{n-1} \) and \( x_{n-2} \); predictor interval = 4h, where \( h = 0.2/\omega_n = \) data-sequence time step.

Figure 2. Predictor output \( \hat{x}_{n+p} \) for input \( x_n \) representing the unit-step response of a 2nd-order system; predictor output formula based on \( x_n, x_{n-1} \) and \( x_{n-2} \); predictor interval = 8h, where \( h = 0.1/\omega_n = \) data-sequence time step.
Next we derive a predictor formula based on \( x_n, x_{n-1}, x_{n-2} \) and \( x_{n-3} \). In this case we solve Eqs. (2), (3) and (4) for \( f_n, f_n', f_n'' \) to order \( h^4 \). This leads directly to the following formulas for \( \hat{f}_n, \hat{f}_n' \) and \( \hat{f}_n'' \):

\[
\hat{f}_n = \frac{11}{6} x_n - \frac{3}{2} x_{n-1} + \frac{1}{3} x_{n-2} - \frac{1}{3} x_{n-3},
\]

(14)

\[
\hat{f}_n' = 2 x_n - 5 x_{n-1} + 4 x_{n-2} - x_{n-3},
\]

(15)

\[
\hat{f}_n'' = x_n - 3 x_{n-1} + 3 x_{n-2} - x_{n-3},
\]

(16)

with the errors in the formulas given by

\[
\begin{align*}
\hat{f}_n - f_n & = -\frac{1}{4} h^3 f_n''', \\
\hat{f}_n' - f_n' & = -\frac{11}{12} h^3 f_n''', \\
\hat{f}_n'' - f_n'' & = -\frac{3}{2} h^3 f_n'''. 
\end{align*}
\]

(17)

(18)

(19)

The prediction formula for \( \hat{x}_{n+p} \) becomes

\[
\hat{x}_{n+p} = x_n + p h f_n + \frac{1}{2} (p h)^2 f_n' + \frac{1}{6} (p h)^3 f_n''.
\]

(20)

From Eqs. (1) and (17) through (20) the predictor error is given approximately by

\[
\hat{x}_{n+p} - x_{n+p} \approx \left[ \frac{1}{4} p + \frac{11}{24} p^2 + \frac{1}{4} p^3 + \frac{1}{24} p^4 \right] h^4 f_n'''.
\]

(21)

When the data sequence \( \{x_n\} \) is a sinusoid given by \( x_n = \sin(\omega n h) \), \( f_n''' \) in Eq. (21) is equal to \( \omega^4 \sin(\omega n h) \). For \( p = 4 \) and \( \omega h = 0.2 \) this results in a steady-state sinusoidal error amplitude of approximately 0.078, compared with the amplitude of 0.16 noted earlier for the case where the predictor method is based on \( x_n, x_{n-1} \) and \( x_{n-2} \). When the predictor input data-sequence \( \{x_n\} \) represents the output response of a second-order linear system, i.e., the case considered in Figure 1, the initial startup transient errors associated with the third-order predictor based on \( x_n, x_{n-1}, x_{n-2} \) and \( x_{n-3} \) are considerably larger than the startup errors shown in Figure 1 for the second-order method based on \( x_n, x_{n-1} \) and \( x_{n-2} \). However, following the initial startup errors, subsequent dynamic errors for the third-order predictor are significantly smaller in magnitude than the errors for the second-order predictor method.

To derive a predictor formula based on \( x_n, x_{n-1}, x_{n-2}, x_{n-3} \) and \( x_{n-4} \). Eqs. (2), (3), (4) and (5) are solved for \( f_n, f_n', f_n'' \) and \( f_n''' \) to order \( h^5 \). This leads directly to the formulas shown in Table 1, which summarizes the predictor formulas of order 2 through 5 when the predictor is based on \( x_n \) and past values of \( x_n \).

3. Predictors Based on \( x_n, x_{n-1}, f_n-1, f_n-2, f_n-3, \ldots \)

When \( x_n \) represents a state variable, which will more often than not be the case in a real-time simulation, the time derivative \( f_n-1 \) will also be available when the data point \( x_n \) is calculated. Then it is better to use a predictor formula based on \( x_n, x_{n-1} \) and its time derivatives \( f_n-1, f_n-2, f_n-3, \ldots \), rather than more past values of \( x_n \). In this case Eq. (2), in addition to the Taylor series formulas for \( f_n-1, f_n-2, f_n-3, \ldots \), written in terms of \( f_n, f_n', f_n'', \ldots \), is used to derive formulas for \( f_n, f_n', f_n'', \ldots \), as needed in the coefficients of the Taylor series equation for the predictor output \( \hat{x}_{n+p} \). Table 2 summarizes the predictor formulas of order 2 through 5, along with the approximate predictor errors when the predictor is based on \( x_n, x_{n-1} \) and past values of its time derivative \( f_n \).
Table 1

Predictors based on \( x_n, x_{n-1}, x_{n-2}, x_{n-3}, \ldots \)

\( \{x_n\} = \) predictor input data sequence; \( \{x_{n+p}\} = \) predictor output data sequence; \( x'_n = f_n \)

2nd-order predictor formula: \( \hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2} (ph)^2 \hat{f}_n'' \), where

\[
h_{\hat{f}}_n = \frac{3}{2} x_n - 2 x_{n-1} + \frac{1}{2} x_{n-2}, \quad h^2 \hat{f}_n'' = x_n - 2 x_{n-1} + x_{n-2}
\]

Predictor output error = \( \hat{x}_{n+p} - x_{n+p} = \left[ \frac{1}{3} p + \frac{1}{2} p^2 + \frac{1}{6} p^3 \right] h^3 \hat{f}_n'' \)

3rd-order predictor formula: \( \hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2} (ph)^2 \hat{f}_n'' + \frac{1}{6} (ph)^3 \hat{f}_n''' \), where

\[
h_{\hat{f}}_n = \frac{11}{6} x_n - 3 x_{n-1} + \frac{5}{3} x_{n-2} - \frac{1}{3} x_{n-3}, \quad h^2 \hat{f}_n'' = 2 x_n - 5 x_{n-1} + 4 x_{n-2} - x_{n-3},
\]

\[
h^3 \hat{f}_n''' = x_n - 3 x_{n-1} + 3 x_{n-2} - x_{n-3}
\]

Predictor output error = \( \hat{x}_{n+p} - x_{n+p} = \left[ \frac{1}{4} p + \frac{11}{24} p^2 + \frac{1}{4} p^3 + \frac{1}{24} p^4 \right] h^4 \hat{f}_n'''' \)

4th-order predictor formula: \( \hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2} (ph)^2 \hat{f}_n'' + \frac{1}{6} (ph)^3 \hat{f}_n''' + \frac{1}{24} (ph)^4 \hat{f}_n'''' \), where

\[
h_{\hat{f}}_n = \frac{25}{12} x_n - 4 x_{n-1} + 3 x_{n-2} - \frac{4}{3} x_{n-3} + \frac{4}{3} x_{n-4}, \quad h^2 \hat{f}_n'' = \frac{35}{12} x_n - \frac{26}{3} x_{n-1} + \frac{19}{2} x_{n-2} - \frac{14}{3} x_{n-3} + \frac{11}{12} x_{n-4},
\]

\[
h^3 \hat{f}_n''' = \frac{5}{2} x_n - 9 x_{n-1} + 12 x_{n-2} - 7 x_{n-3} + \frac{3}{2} x_{n-4}, \quad h^4 \hat{f}_n'''' = x_n - 4 x_{n-1} + 6 x_{n-2} - 4 x_{n-3} + x_{n-4}
\]

Predictor output error = \( \hat{x}_{n+p} - x_{n+p} = \left[ \frac{1}{5} p + \frac{5}{12} p^2 + \frac{7}{24} p^3 + \frac{1}{12} p^4 + \frac{1}{120} p^5 \right] h^5 \hat{f}_n''''' \)

5th-order predictor formula: \( \hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2} (ph)^2 \hat{f}_n'' + \frac{1}{6} (ph)^3 \hat{f}_n''' + \frac{1}{24} (ph)^4 \hat{f}_n'''' + \frac{1}{120} (ph)^5 \hat{f}_n''''' \), where

\[
h_{\hat{f}}_n = \frac{137}{60} x_n - 5 x_{n-1} + 5 x_{n-2} - \frac{10}{3} x_{n-3} + \frac{5}{4} x_{n-4} - \frac{1}{5} x_{n-5},
\]

\[
h^2 \hat{f}_n'' = \frac{15}{4} x_n - \frac{77}{6} x_{n-1} + \frac{107}{6} x_{n-2} - 13 x_{n-3} + \frac{61}{12} x_{n-4} - \frac{5}{6} x_{n-5},
\]

\[
h^3 \hat{f}_n''' = \frac{17}{4} x_n - \frac{71}{4} x_{n-1} + \frac{59}{2} x_{n-2} - \frac{49}{2} x_{n-3} + \frac{41}{4} x_{n-4} - \frac{7}{4} x_{n-5},
\]

\[
h^4 \hat{f}_n'''' = 3 x_n - 14 x_{n-1} + 26 x_{n-2} - 24 x_{n-3} + 11 x_{n-4} - 2 x_{n-5},
\]

\[
h^5 \hat{f}_n''''' = x_n - 5 x_{n-1} + 10 x_{n-2} - 10 x_{n-3} + 5 x_{n-4} - x_{n-5},
\]

Predictor output error = \( \hat{x}_{n+p} - x_{n+p} = \left[ \frac{1}{5} p + \frac{137}{360} p^2 + \frac{15}{48} p^3 + \frac{17}{144} p^4 + \frac{5}{240} p^5 + \frac{1}{720} p^6 \right] h^6 \hat{f}_n'''''' \)
Table 2

Predictors based on \(x_n, x_{n-1}, f_{n-1}, f_{n-2}, f_{n-3}, \ldots\)

\(\{x_n\} = \) predictor input data sequence; \(\{x_{n+p}\} = \) predictor output data sequence; \(x'_n = f_n\)

2nd-order predictor formula: \(\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)^2\hat{f}'_n\), where

\(\hat{f}_n = 2x_n - 2x_{n-1} - hf_{n-1}\), \(\hat{f}'_n = 2x_n - 2x_{n-1} - 2hf_{n-1}\),

Predictor output error \(= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{1}{6} p + \frac{1}{3} p^2 + \frac{1}{6} p^3\right]h^3f''\)

3rd-order predictor formula: \(\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)^2\hat{f}'_n + \frac{1}{6}(ph)^3\hat{f}''_n\), where

\(\hat{f}_n = \frac{12}{5}x_n - \frac{12}{5}x_{n-1} - \frac{8}{5}hf_{n-1} + \frac{1}{5}h^2f_{n-2}\), \(\hat{f}'_n = \frac{18}{5}x_n - \frac{18}{5}x_{n-1} - \frac{22}{5}hf_{n-1} + \frac{4}{5}hf_{n-2}\), \(\hat{f}''_n = \frac{12}{5}x_n - \frac{12}{5}x_{n-1} - \frac{18}{5}hf_{n-1} + \frac{6}{5}hf_{n-2}\)

Predictor output error \(= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{1}{10} p + \frac{29}{120} p^2 + \frac{11}{60} p^3 + \frac{1}{24}\right]h^4f'''

4th-order predictor formula: \(\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)^2\hat{f}'_n + \frac{1}{6}(ph)^3\hat{f}''_n + \frac{1}{24}(ph)^4\hat{f}'''_n\), where

\(\hat{f}_n = \frac{8}{3}x_n - \frac{8}{3}x_{n-1} - \frac{19}{9}hf_{n-1} + \frac{5}{9}h^2f_{n-2} - \frac{1}{9}h^3f_{n-3}\), \(\hat{f}'_n = \frac{44}{9}x_n - \frac{44}{9}x_{n-1} - \frac{371}{54}hf_{n-1} + \frac{68}{27}hf_{n-2} - \frac{29}{54}hf_{n-3}\), \(\hat{f}''_n = \frac{16}{3}x_n - \frac{16}{3}x_{n-1} - \frac{83}{9}hf_{n-1} + \frac{46}{9}hf_{n-2} - \frac{11}{9}hf_{n-3}\), \(\hat{f}'''_n = \frac{8}{3}x_n - \frac{8}{3}x_{n-1} - \frac{46}{3}hf_{n-1} + \frac{32}{9}hf_{n-2} - \frac{10}{9}hf_{n-3}\), \(\hat{f}''''_n = \frac{8}{3}x_n - \frac{8}{3}x_{n-1} - \frac{46}{3}hf_{n-1} + \frac{32}{9}hf_{n-2} - \frac{10}{9}hf_{n-3}\)

Predictor output error \(= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{19}{270} p + \frac{307}{1620} p^2 + \frac{571}{3240} p^3 + \frac{212}{3240} p^4 + \frac{1}{120} p^5\right]h^5f''''

5th-order predictor formula: \(\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)^2\hat{f}'_n + \frac{1}{6}(ph)^3\hat{f}''_n + \frac{1}{24}(ph)^4\hat{f}'''_n + \frac{1}{120}(ph)^5\hat{f}''''_n\), where

\(\hat{f}_n = \frac{720}{251}x_n - \frac{720}{251}x_{n-1} - \frac{646}{251}hf_{n-1} + \frac{264}{251}hf_{n-2} - \frac{106}{251}hf_{n-3} + \frac{19}{251}hf_{n-4}\), \(\hat{f}'_n = \frac{1500}{251}x_n - \frac{1500}{251}x_{n-1} - \frac{14099}{1506}hf_{n-1} + \frac{1303}{251}hf_{n-2} - \frac{1111}{251}hf_{n-3} + \frac{307}{251}hf_{n-4}\), \(\hat{f}''_n = \frac{1800}{251}x_n - \frac{1800}{251}x_{n-1} - \frac{3874}{251}hf_{n-1} + \frac{3672}{251}hf_{n-2} - \frac{2022}{251}hf_{n-3} + \frac{424}{251}hf_{n-4}\), \(\hat{f}'''_n = \frac{720}{251}x_n - \frac{720}{251}x_{n-1} - \frac{1650}{251}hf_{n-1} + \frac{1770}{251}hf_{n-2} - \frac{1110}{251}hf_{n-3} + \frac{270}{251}hf_{n-4}\), \(\hat{f}''''_n = \frac{2100}{251}x_n - \frac{2100}{251}x_{n-1} - \frac{8119}{502}hf_{n-1} + \frac{6309}{502}hf_{n-2} - \frac{2961}{502}hf_{n-3} + \frac{571}{502}hf_{n-4}\), \(\hat{f}'''''_n = \frac{81}{1506} p + \frac{3133}{20080} p^2 + \frac{5965}{36144} p^3 + \frac{473}{6024} p^4 + \frac{1031}{60240} p^5 + \frac{1}{720} p^6\)

Predictor output error \(= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{81}{1506} p + \frac{3133}{20080} p^2 + \frac{5965}{36144} p^3 + \frac{473}{6024} p^4 + \frac{1031}{60240} p^5 + \frac{1}{720} p^6\right]h^6f'''''

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4. Predictors Using $x_n, f_n, f_{n-1}, f_{n-2}, f_{n-3}, \ldots$

Not infrequently in a dynamic simulation both the output data-point $x_n$ and its time derivative $f_n$ are state variables. For example, this may be the case when the system being simulated involves a double integration of acceleration to compute velocity and displacement. When this is true, both $x_n$ and $f_n$ are available for calculating $\hat{x}_{n+p}$. The predictor formula can then be based on $x_n, f_n, f_{n-1}, f_{n-2}, f_{n-3}, \ldots$. Formulas for $\hat{f}_n, \hat{f}'_n, \hat{f}''_n, \ldots$ are again derived from Taylor series representing $f_{n-1}, f_{n-2}, f_{n-3}, \ldots$ in terms of $f_n, f'_n, f''_n, \ldots$. Table 3 summarizes the predictor formulas of order 2 through 5, along with the approximate predictor errors. It should be noted that whenever the prediction index $p = 1$, the formulas for $\hat{x}_{n+p}$ given in Table 3 reduce to the formula for $x_{n+1}$ in the Adams-Bashforth predictor integration algorithm of order 2.

Table 3

Predictors based on $x_n, f_n, f_{n-1}, f_{n-2}, f_{n-3}, \ldots$

{\{x_n\} = predictor input data sequence; \{x_{n+p}\} = predictor output data sequence; \prime f_n $\equiv$ \prime f_n$

2nd-order predictor formula: $\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)f'_n$, where $h^2f'_n = \frac{3}{2}f_n - \frac{1}{2}f_{n-1}$

Predictor output error $= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{1}{4}p^2 + \frac{1}{6}p^3\right]h^3f''_n$

3rd-order predictor formula: $\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)f'_n + \frac{1}{6}(ph)f''_n$, where $h^2f'_n = \frac{3}{2}hf_n - 2hf_{n-1} + \frac{1}{2}hf_{n-2}$, $h^3f''_n = hf_n - 2hf_{n-1} + hf_{n-2}$

Predictor output error $= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{1}{6}p^2 + \frac{1}{6}p^3 + \frac{1}{24}p^4\right]h^4f'''_n$

4th-order predictor formula: $\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)f'_n + \frac{1}{6}(ph)f''_n + \frac{1}{24}(ph)f'''_n$, where $h^2f'_n = \frac{11}{6}hf_n - 3hf_{n-1} + \frac{3}{2}hf_{n-2} - \frac{1}{3}hf_{n-3}$, $h^3f''_n = 2hf_n - 5hf_{n-1} + 4hf_{n-2} - hf_{n-3}$

Predictor output error $= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{1}{8}p^2 + \frac{11}{72}p^3 + \frac{1}{16}p^4 + \frac{1}{120}p^5\right]h^5f''''_n$

5th-order predictor formula: $\hat{x}_{n+p} = x_n + ph\hat{f}_n + \frac{1}{2}(ph)f'_n + \frac{1}{6}(ph)f''_n + \frac{1}{24}(ph)f'''_n + \frac{1}{120}(ph)f''''_n$, where $h^2f'_n = \frac{25}{12}hf_n - 4hf_{n-1} + 3hf_{n-2} - \frac{4}{3}hf_{n-3} + \frac{1}{4}hf_{n-4}$,

$h^3f''_n = \frac{35}{12}hf_n - \frac{26}{3}hf_{n-1} + \frac{19}{2}hf_{n-2} - \frac{14}{3}hf_{n-3} + \frac{11}{12}hf_{n-4}$,

$h^4f'''_n = \frac{5}{2}hf_n - 9hf_{n-1} + 12hf_{n-2} - 7hf_{n-3} + \frac{3}{2}hf_{n-4}$, $h^5f''''_n = hf_n - 4hf_{n-1} + 6hf_{n-2} - 4hf_{n-3} + hf_{n-4}$

Predictor output error $= \hat{x}_{n+p} - x_{n+p} = -\left[\frac{1}{10}p^2 + \frac{5}{36}p^3 + \frac{7}{96}p^4 + \frac{1}{60}p^5 + \frac{1}{720}p^6\right]h^6f''''''_n$
In order to compare the relative accuracy of the different predictor methods summarized in Tables 1, 2 and 3 we rewrite the formulas for the approximate output error for a predictor method of order $k$ in the following form:

$$\text{Predictor output error} = \hat{x}_{n+p} - x_{n+p} = -f_k(p)(ph)^{k+1} \frac{d^k f_n}{dt^k}, \quad (22)$$

where for predictor formulas of order 2, 3, 4 and 5 in Table 1 the function $f_k(p)$ is given by

$$f_2(p) = \left[ \frac{1}{3p^2} + \frac{1}{2p} + \frac{1}{6} \right], \quad f_3(p) = \left[ \frac{1}{4p^3} + \frac{11}{24p^2} + \frac{1}{4p} + \frac{1}{24} \right], \quad (23)$$

$$f_4(p) = \left[ \frac{1}{5p^4} + \frac{5}{12p^3} + \frac{7}{24p^2} + \frac{1}{12p} + \frac{1}{120} \right], \quad (24)$$

$$f_5(p) = \left[ \frac{1}{6p^5} + \frac{137}{360p^4} + \frac{15}{48p^3} + \frac{17}{144p^2} + \frac{5}{240p} + \frac{1}{720} \right]. \quad (25)$$

In the same way formulas for the function $f_k(p)$ can be written for the predictor methods of order 2, 3, 4 and 5 given in Table 2. Thus

$$f_2(p) = \left[ \frac{1}{6p^2} + \frac{1}{3p} + \frac{1}{6} \right], \quad f_3(p) = \left[ \frac{1}{10p^3} + \frac{29}{120p^2} + \frac{11}{60p} + \frac{1}{24} \right], \quad (26)$$

$$f_4(p) = \left[ \frac{19}{270p^4} + \frac{307}{1620p^3} + \frac{571}{3240p^2} + \frac{212}{3240p} + \frac{1}{120} \right], \quad (27)$$

$$f_5(p) = \left[ \frac{81}{1506p^5} + \frac{3133}{20080p^4} + \frac{5965}{36144p^3} + \frac{473}{6024p^2} + \frac{1031}{60240p} + \frac{1}{720} \right]. \quad (28)$$

For the predictor methods of order 2, 3, 4 and 5 in Table 3, the function $f_k(p)$ is given by

$$f_2(p) = \left[ \frac{1}{4p} + \frac{1}{6} \right], \quad f_3(p) = \left[ \frac{1}{6p^2} + \frac{1}{6p} + \frac{1}{24} \right], \quad (29)$$

$$f_4(p) = \left[ \frac{1}{8p^3} + \frac{11}{72p^2} + \frac{1}{16p} + \frac{1}{120} \right], \quad (30)$$

$$f_5(p) = \left[ \frac{1}{10p^4} + \frac{5}{36p^3} + \frac{7}{96p^2} + \frac{1}{60p} + \frac{1}{720} \right]. \quad (31)$$

With the order $k = 2$ for each of the three predictor methods, Figure 3 shows plots of the function $f_2(p)$ given in Eqs. (23), (26) and (29) versus the predictor index $p$. Similar plots of $f_k(p)$ for $k = 3, 4$ and 5, as obtained using Eqs. (23) through (29), are shown in Figures 4, 5 and 6, respectively. The figures clearly show that $\hat{x}_{n+p}$ based on $f_n, f_{n-1}, f_{n-2}, \ldots$, which can be used when both $x_n$ and $f_n (= x'_n)$ are state variables, always yields the most accurate prediction. Prediction with $\hat{x}_{n+p}$ based on $x_n, x_{n-1}, f_{n-1}, f_{n-2}, \ldots$, which can be used when only $x_n$ is a state variable, yields the next most accurate results. Prediction with $\hat{x}_{n+p}$ based on $x_n, x_{n-1}, x_{n-2}, \ldots$, which must be used when $x_n$ is not a state variable, gives the least accurate results.
Figure 3. Comparative accuracy of three different second-order predictors.

Figure 4. Comparative accuracy of three different third-order predictors.
Figure 5. Comparative accuracy of three different fourth-order predictors.

Figure 6. Comparative accuracy of three different fifth-order predictors.
5. Other Methods for Obtaining Prediction

In addition to the use of the predictor formulas described above in Sections 2, 3 and 4, there are several alternate methods to achieve a time lead in real-time simulation. The first method can be used when both \( x_n \) and its time derivative \( x'_n \) are state variables in a simulation. We recall that this is a prerequisite for the predictor of Section 4, based on \( x_n, f_n, f_{n-1}, f_{n-2}, \ldots \). When using a single-pass integration method such as Adams-Bashforth, during the \( n \)th integration frame we can integrate \( x_{n+1} \) to obtain \( x_{n+2} \) at the end of the \( n \)th frame. This results in the availability of the state-variable \( x \) one entire integration step before it occurs in real time. This is equivalent to using a predictor formula for \( x_{n+p} \) in Table 3 of the same order as the Adams-Bashforth integration method used in the simulation, with the predictor index \( p \) set equal to 1.

A second method of obtaining a pure time lead in a real-time simulation is to utilize a numerical integration step-size \( h \) which is larger than the processor execution time for one integration step. In past years, when digital processors were much slower than current processors, it was customary in a real-time simulation to set the step-size \( h \) equal to the processor execution time for a single integration step, thus minimizing the dynamic errors caused by the numerical simulation. With the much higher processor speeds currently available, this procedure may often not be required. For example, if the integration step size \( h \) is set equal to twice the processor execution time for each integration frame, the simulation output \( x_n \) is available \( h/2 \) seconds ahead of real time. This then becomes an exact numerical prediction of \( h/2 \) seconds for \( x_n \). This in turn can be used to compensate for the half-frame delay associated with D to A converters using zero-order hold extrapolation. On the other hand, if Eq. (22) for the predictor output error \( \hat{x}_{n+p} - x_{n+p} \) is examined, along with the \( f_k(p) \) functions shown in Figures 3, 4, 5 and 6, it is apparent that for a given prediction time-interval \( ph \), it is better to maximize the prediction index \( p \). This in turn is achieved by minimizing the time step \( h \) associated with the predictor input data-sequence \( \{x_n\} \), which is achieved by setting the integration step used for the real-time simulation that generates \( x_n \) equal to the processor execution time for each step.

6. Summary

In this paper we have described a number of predictor formulas which can be utilized in a simulation to provide real-time outputs with a time lead of \( ph \) seconds, as needed to compensate for any time delays occurring elsewhere in the simulation. The predictor formulas are summarized in Tables 1, 2 and 3, which include formulas for the approximate error in predicted output \( \hat{x}_{n+p} \). The relative accuracy of different orders of predictor formulas is presented in Figures 3 through 6. When a predictor input \( x_n \) contains discontinuities, the simulation results in Figures 1 and 2 remind us that there is no method that can produce an output \( \hat{x}_{n+p} \) which represents an accurate prediction prior to the time at which the discontinuity occurs. The formulas for the approximate predictor output errors in Tables 1, 2 and 3 demonstrate that for a given prediction time-interval \( ph \), the predictor output error will be minimized when the time step \( h \) of the predictor input data sequence \( \{x_n\} \) is minimized, since this maximizes the prediction index \( p \). This suggests that in a real-time simulation requiring the use of predictor formulas, the integration time-step \( h \) used for numerical integration in the simulation should be set equal to the maximum processor execution time required for each integration step.